JOURNAL OF COMPUTATIONAL PHYSICS 91, 486–489 (1990)

Note

Conservation and Breaking of Mirror Symmetry in a Numerical Simulation of Vortex Flow

Growth and bifurcation of a symmetry broken state out of a symmetric one as an external control parameter is varied across a threshold value occurs in many physical systems. Consider a forward bifurcation where the symmetry broken state is stable beyond the threshold and the symmetric one is unstable. Then the former will grow and the latter will decay only if there are perturbations that break the considered symmetry. In this note we investigate the conditions under which numerical rounding errors of a finite difference MAC [1] simulation provide perturbations that break a mirror symmetry.

As an example we consider the experimentally, analytically, and numerically well-investigated axisymmetric vortex flow in the rotating Couette system [3–7] between two concentric cylinders of inner radius r_1 and outer radius r_2 with height H being similar to the gapwidth $d = r_2 - r_1$. The inner one rotates with constant angular velocity Ω, the outer one is at rest, and a stationary collar closes the gap at the cylinder ends, $z = \pm H/2$. Below a threshold angular velocity Ω_c, defining a threshold Reynolds number $R_c = \Omega_c r_1 d/\nu$ ($\simeq 128$ for $H = 1.05\,d$ and $\eta = r_1/r_2 \simeq 0.5$), where ν is the kinematic viscosity of the fluid, there are two compressed vortices that are mirror images of each other. The mirror plane is the plane $z = 0$. This flow state as shown in Fig. 1a is unique in the sense that it always develops under quasistatic increase of the control parameter R from zero. Above R_c the mirror symmetric flow while remaining a solution of the Navier–Stokes equations (NSE) is unstable and a vortex flow is stable that breaks the mirror symmetry of the system. It consists of two vortices (that are still axisymmetric) of different size—cf. Fig. 1b. With increasing R one of them grows in size on cost of the other. Which of the two grows is determined by symmetry breaking perturbations or initial values. Note that the results of numerical simulations and experiments on all these symmetry, stability, and bifurcation properties agree with each other.

Following SOLA [2] we have integrated the axisymmetric Navier–Stokes equations for the velocity $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{e}_z$ and the pressure p on the twodimensional $r - z$ cross section between the cylinders. Here $\mathbf{e}$ are radial, azimuthal, and axial unit vectors. The discretized MAC version of the NSE is as its continuous counterpart mirror symmetric. Thus starting from *symmetric* initial conditions, there are only the rounding errors that can possibly introduce mirror symmetry breaking. We

0021-9991/90 $3.00

shall demonstrate this as a representative example at the dissipative term $D = \partial_z^2 u$ entering the MAC equations for the radial velocity u in the form

$$D(r, z) = \frac{1}{(\Delta z)^2} [A - 2B + C]. \tag{1}$$

Here

$$A = u(r, z + \Delta z); \qquad B = u(r, z); \qquad C = u(r, z - \Delta z) \tag{2}$$

denote values of u on axially neighboring grid points at some time t. At the mirror image of the position r, z the dissipation term reads

$$D(r, -z) = \frac{1}{(\Delta z)^2} [C' - 2B' + A'], \tag{3}$$

where the primed quantities denote the values of u at respective mirror positions (cf. Fig. 2).

Without rounding errors one obtains $D(r, z) = D(r, -z)$ if $u = u'$, i.e., if $A = A'$,

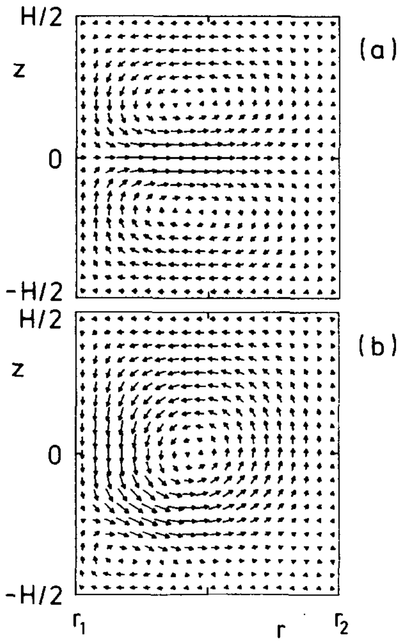

FIG. 1. Vortex flow field in the $r-z$ cross section of the gap between concentric cylinders of the rotating Couette system: (a) Unstable mirror symmetric vortices above threshold ($R_c \simeq 128$ for $H = 1.05\,d$, $\eta \simeq 0.507$). Below threshold this state is stable. (b) Stable symmetry broken flow above threshold R_c.

$B = B'$, $C = C'$. Thus an initially symmetric field would indeed remain symmetric. Rounding errors, however, cause (computer addition is not associative)

$$(A - 2B) + C \neq (C - 2B) + A, \tag{4}$$

leading to $D(r, z) \neq D(r, -z)$ and thus to mirror symmetry breaking.

Now consider as an alternative the mirror preserving computer instruction

$$D(r, z) = \frac{1}{(\Delta z)^2} [(A - B) - (B - C)], \tag{5}$$

where $\partial_z^2 u$ is evaluated consecutively out of the first derivatives, $\partial_z^2 u = \partial_z(\partial_z u)$. Its roundoff errors are mirror symmetric since

$$D(r, -z) = \frac{1}{(\Delta z)^2} [(C' - B') - (B' - A')] \tag{6}$$

is equal to $D(r, z)$ (computer addition is commutative) if $A = A'$, $B = B'$, $C = C'$. Formulating in a similar manner all finite difference expressions of the MAC algorithm in this mirror symmetry preserving way (5) we in fact could not generate the symmetry broken flow state of Fig. 1b when we started from *symmetric* initial conditions. (Starting from *asymmetric* initial conditions, however, we naturally got a stationary asymmetric state for $R > R_c$.) With finite difference instructions of type (1) that did not suppress the appearence of mirror symmetry breaking roundoff errors, on the other hand, the code ran for $R > R_c$ into the symmetry broken stable solution and for $R < R_c$ into the symmetric state. Our considerations apply in a similar way also to other flow problems. An example is the unstable stationary mirror symmetric flow of convective rolls in the Rayleigh–Bénard system with a binary fluid mixture. There, for some parameter values, left or right travelling waves that break mirror symmetry are the stable solutions [8].

As a conclusion we remark that in order to obtain stable symmetry broken solutions in MAC simulations of symmetric equations with symmetric boundaries one

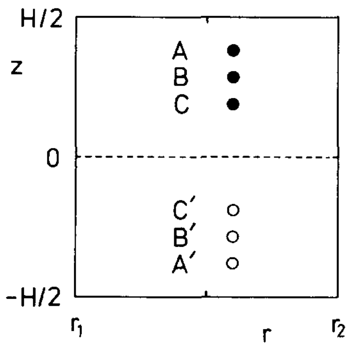

FIG. 2. Grid points of the MAC code for evaluating $\partial_z^2 u$.

either has to start from asymmetric initial conditions or provide symmetry breaking perturbations. The latter can arise from rounding errors when the finite difference expressions are formulated in a particular way. Eliminating these two symmetry breaking mechanisms altogether, the MAC code generates symmetric solutions irrespective of whether they are stable or unstable.

ACKNOWLEDGMENT

This work was supported by Stiftung Volkswagenwerk.

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RECEIVED: April 25, 1989; REVISED: July 31, 1989

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